Instability of Travelling Waves in a Complex System Modelling Fire Spread

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Abstract: A system of coupled partial differential equations which models a complex system of a solid fuel, endothermically pyrolysing to a combustible gas, which in turn exothermically reacts with oxygen, is studied. It is shown that there are travelling wave solutions to the model and there seem to be reasonable parameter values for which this model exhibits oscillatory propagating combustion waves.

Keywords **:** Combustion; fire; modelling

1. INTRODUCTION

Fire modelling has long been an active research topic [Drysdale, 1999; Cox, 1995; Johnson and Miyanishi, 2001] with applications to both urban and rural fires. More recently, there have been several investigations into more comprehensive models of fire behaviour [Grishin, 1984; Weber, 1991; Di Blasi, 1993; Larini et al, 1998; Asensio and Ferragut, 2001]. These models have all involved a solid fuel which undergoes endothermic gasification. The product of the gasification can then react exothermically with oxygen and the heat released can (if sufficient) propagate the fire. All of the above mentioned fire models assumed the existence of a stable, steadily propagating combustion wave, which is to be interpreted as the model representation of a spreading fire. Furthermore, the limited numerical calculations (for selected parameter values) have always converged to such a stable, steadily propagating combustion wave. In view of the findings in studies of mathematical models of gasless combustion [e.g. Bayliss and Matkowsky, 1994; Mercer et al., 1998; Weber et al., 1997] where there seems to always exist parameter values beyond which instabilities are inherent in the model (as distinct from numerical method instabilities which need to be carefully eliminated), it is surprising that the aforementioned workers have not found (or at least commented upon) similar oscillatory or unstable waves in the more complex models.

In this paper, we report on our initial investigations of a simple, yet in some sense, generic model, which encompasses all of the main features required in a complex model of a solid fuel which passes through a gaseous intermediary prior to combustion. The gas/solid coupling, the endothermic pyrolysis and the exothermic gaseous combustion are all included. Unfortunately, other than very limited analytical work, our results must all be obtained numerically. We are able to find travelling combustion waves in our model, in accordance with previous investigators and we have found evidence of oscillatory propagating combustion waves for selected parameter values.

2. MATHEMATICAL MODEL

We begin with the conservation of energy and mass, written out for the solid phase and the gaseous phase. The subscripts refer to the solid phase (s) or the gaseous phase (g) and we assume that the solid fuel cannot diffuse heat or mass, while the gaseous fuel diffuses as usual. There is heat transfer between the phases, modelled with a linear term and pyrolysis of the solid depletes the solid fuel and is a source term for the gaseous fuel. The presentation and solution of the equations is greatly simplified if they are **non-dimensionalised** in a standard way [e.g. following Forbes and Gray, 1998, Mercer et al., 1998; Weber et al., 1997], whence the equations for our model can be written as

$$\begin{aligned}
\frac{\partial u_s}{\partial t} &= -Q_s v_s e^{-\gamma/u_s} + h_s(u_g - u_s) \\
&\quad -\ell_s(u_s - u_a) \\
\frac{\partial v_s}{\partial t} &= -\beta_s v_s e^{-\gamma/u_s} \\
\frac{\partial u_g}{\partial t} &= \nabla^2 u_g + Q_g v_g e^{-1/u_g} - h_g(u_g - u_s) \\
&\quad -\ell_g(u_g - u_a) \\
\frac{\partial v_g}{\partial t} &= \frac{1}{\mathrm{Le}}\nabla^2 v_g - \beta_g g e^{-1/u_g} + \beta_s v_s e^{-\gamma/u_s}
\end{aligned}$$

The main features of the model can be seen from selected terms in the equations. We note that the endothermic pyrolysis has a heat of combustion Q_s, stoichiometric coefficient β_s and activation energy γ. The heat transfer between the gas and solid occurs through terms with constants h_g and h_s. The gaseous exothermic combustion releases energy Q_g with stoichiometric coefficient β_g and activation energy unity (in these non-dimensional units). Heat losses to the environment occur through the terms with parameters ℓ_g and ℓ_s.

The equations for our model are expected to possess travelling wave solutions which would be identified as modelling a spreading fire. It is not mathematically certain that such solutions exist, although such reactive-diffusive systems usually do possess wave-like solutions for at least some parameter values. As we will see, the numerical studies confirm the existence of such wave-like solutions, but the question of stability is much harder to address in a comprehensive manner.

3. OBSERVATIONS AND NUMERICAL RESULTS

We begin by noticing that the system of equations has a limit in which it reduces exactly to the gaseous combustion model considered in Weber et al. [1997]. This is for the particular parameter values $Q_g = 1$, $Q_s = 0$, $\beta_s = 0$, $\mathrm{Le} \to \infty$, $\ell_s = 0$, $\ell_g = 0$; whereupon $u_s = u_g$ (and hence h_s and h_g are irrelevant) and v_s is constant. As Weber et al. [1997] show, the model in this particular case exhibits an instability in the travelling wave solution when β_g is greater than 6.5 and the instability appears to be a period doubling route to chaos, similar to that identified earlier by Bayliss and Matkowsky [1994] for a model of gasless combustion with delta function kinetics. For this reason we are lead to suspect that the model as described in the equations will indeed possess travelling combustion wave solutions and that instabilities may manifest themselves in certain parameter regions. Actually, our attempts to verify this with the parameter values given above were fraught with difficulties due to the singular nature of the limit. Hence, we chose instead to begin with milder values for the parameters and quickly chanced upon a set which surprised us by showing a very clear oscillation in the temperatures, fuel consumption and consequently the speed of propagation.

Figures 1–5 show the results of numerically solving the model in a particular case when the system is similar to that introduced by Please et al. [2001] and for which the parameter values are $\mathrm{Le} = 1$, $\beta_g = \beta_s = 6$, $Q_s = 0.1$, $Q_g = 1.0$, $h_s = h_g = 0.1$, $\gamma = 1.0$, $\ell_s = \ell_g = 0$. Naturally, this is the case with simple stoichiometry and sufficient heat exchange between the solid and gas to reduce to a single temperature, as well as no overall heat losses. The numerical solution method consisted of finite differences in space and adaptive time stepping as described in more detail by Mercer et al. [1998].

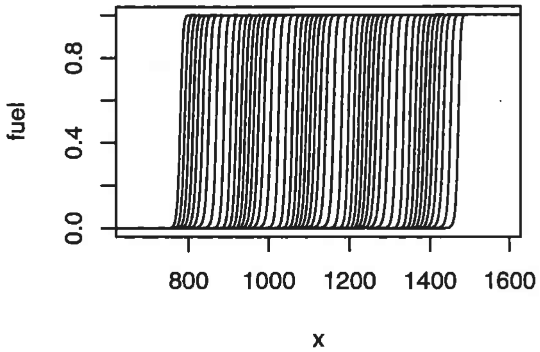

Figure 1: Solid Fuel Contours

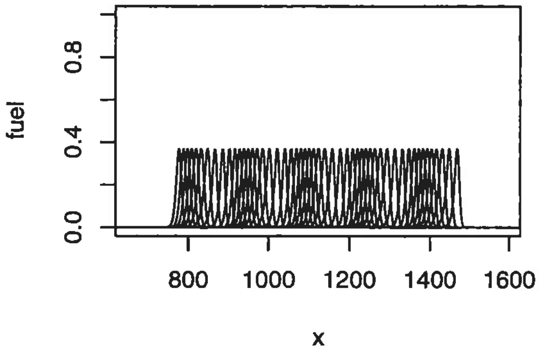

Figure 2: Pyrolysed Fuel Contours

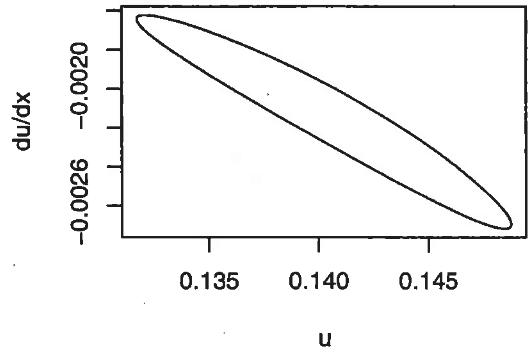

Figure 3: Phase Plane

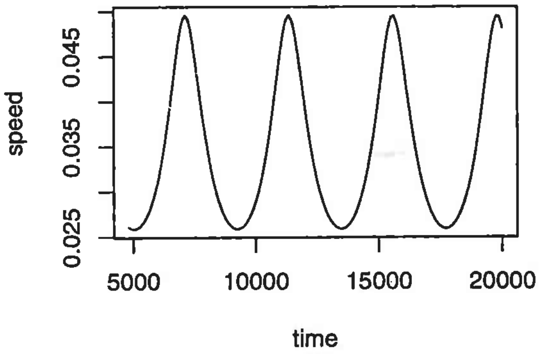

Figure 4: Speed variation

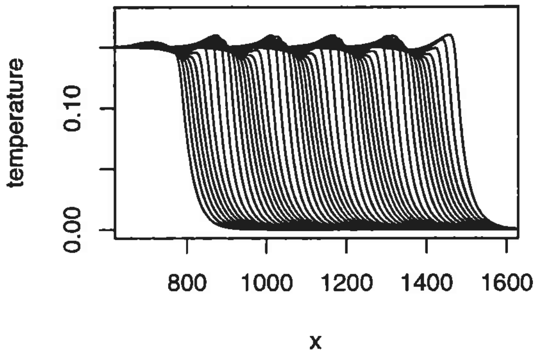

Figure 5: System temperature

Note that all variables presented are nondimensional as per the equations for this model given earlier.

Clearly, there is evidence of an interesting cycle in the pyrolysis, combustion and propagation. This is best seen in the speed graph, where there appears to be a slow build up followed by a burst and this cycles over a well defined time scale. This result was quite unexpected and is not at all evident from any cursory analysis of the model. The precise mathematical reason(s) for this instability are currently under active investigation using a variety of analytical and semi-analytical methods.

While the precise details of this sort of instability are of scientific interest only to specialists in this type of mathematical modelling, the implications are of much broader significance. In particular, it is important to recognise that models of continuous processes can display unexpected complexity; be it in the context of combustion, meteorological or other processes. The larger, as yet unanswered, question is whether or not the complexity found in the models is of real significance for the true physical process of interest.

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